OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

Specimen Test One – Issued March 2009

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

NOTE: Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.

NAME:

TEST CENTRE:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[ ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ____________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

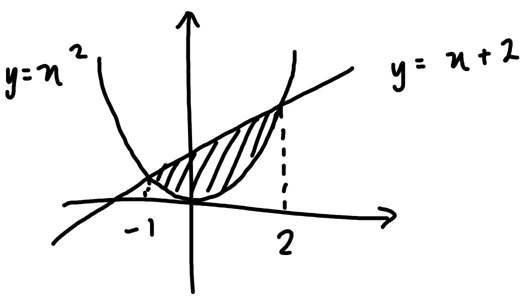
1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A–J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The area of the region bounded by the curves $y = x^2$ and $y = x + 2$ equals

(a) $\frac{7}{3}$ (b) $\frac{7}{2}$ ✓(c) $\frac{9}{2}$ (d) $\frac{11}{2}$



$x^2 = x + 2$

$x^2 - x - 2 = 0$

$y = x^2$ and $y = x + 2$ meet a $x = 2$, $x = 1$

when $x = 2$, $y = 4$

when $x = -1$, $y = 1$

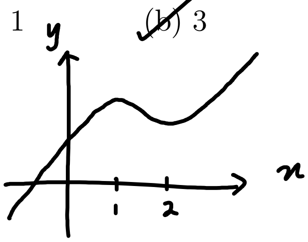
$$\text{Area required} = \int_{-1}^{2} x + 2\,dx - \int_{-1}^{2} x^2\,dx$$
$$= \left(\frac{4+1}{2}\right)(2+1) - \left[\frac{x^3}{3}\right]_{-1}^{2}$$
$$= \frac{5}{2} \times 3 - \left[\frac{8}{3} - \left(-\frac{1}{3}\right)\right]$$
$$= \frac{15}{2} - 3$$
$$= \frac{9}{2}$$

B. The smallest value of the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

in the range $0 \le x \le 2$ is

(a) 1 ✓(b) 3 (c) 5 (d) 7



$f(x) = 2x^3 - 9x^2 + 12x + 3$

$f'(x) = 6x^2 - 18x + 12 = 0$

$6(x-2)(x-1) = 0$

turning points at $x = 1$ (max) and $x = 2$ (min)

Lowest point in the range $0 \le x \le 2$ either at $x = 0$ or $x = 2$

At $x = 2$, $f(x) = 7$. At $x = 0$, $f(x) = 3$

$\therefore$ smallest value of $f(x)$ is 3

C. What is the reflection of the point $(3, 4)$ in the line $3x + 4y = 50$?

(a) $(9, 12)$ ✓ (b) $(6, 8)$ (c) $(12, 16)$ (d) $(16, 12)$

gradient $= \frac{-3}{4}$

$\therefore$ gradient of normal $= \frac{4}{3}$

$y = \frac{4}{3}x + c$

$4 = 4 + c \quad c = 0$

when $y = \frac{4}{3}x$

$3x + 4\left(\frac{4}{3}x\right) = 50$

$x = 6 \quad y = 8$

$(6, 8)$ is the midpoint of $(3, 4)$ to the reflection

$\therefore$ reflection is $(9, 12)$

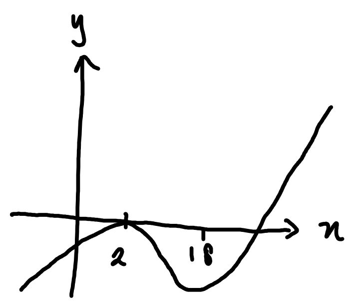
D. The equation $x^3 - 30x^2 + 108x - 104 = 0$

(a) no real roots;

(b) exactly one real root;

(c) three distinct real roots;

(d) a repeated root. ✓



Let $y = x^3 - 30x^2 + 108x - 104 = 0$

$\frac{dy}{dx} = 3x^2 - 60x + 108 = 0$

$3(x^2 - 20x + 36) = 0$

$3(x - 18)(x - 2) = 0$

turning points at $x = 18,\ x = 2$

when $x = 2$, $y = 0$. A turning point occurs when $y = 0$

$\therefore$ there is a repeated root at $x = 2$

E. The fact that

$$6 \times 7 = 42,$$

is a counter-example to which of the following statements?

(a) the product of any two odd integers is odd;
(b) if the product of two integers is not a multiple of 4 then the integers are not consecutive;
(c) if the product of two integers is a multiple of 4 then the integers are not consecutive;
(d) any even integer can be written as the product of two even integers.

Both 6 and 7 are integers and their product (42) is not a multiple of 4, fulfilling the initial conditions of statement b. However, 6 and 7 are consecutive, meaning b is proved to be incorrect by $6 \times 7 = 42$

F. How many values of x satisfy the equation

$$2\cos^2 x + 5\sin x = 4$$

in the range $0 \leqslant x < 2\pi$?

(a) 2 (b) 4 (c) 6 (d) 8

$2\cos^2 x + 5\sin x = 4$

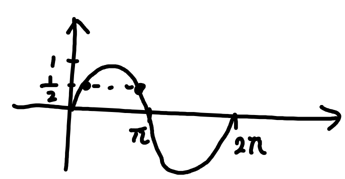
$2(1 - \sin^2 x) + 5\sin x = 4$ $(\cos^2 x + \sin^2 x = 1)$

$2\sin^2 x - 5\sin x + 2 = 0$

$(2\sin x - 1)(\sin x - 2) = 0$

$\sin x = \frac{1}{2}$ $\sin x = 2$
no solutions

2 solutions in the range required

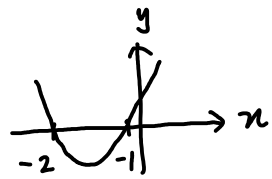


Turn Over

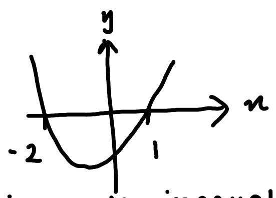
G. The inequalities $x^2 + 3x + 2 > 0$ and $x^2 + x < 2$, are met by all x in the region:

(a) $x < -2$;
(b) $-1 < x < 1$;
(c) $x > -1$;
(d) $x > -2$.

$x^2 + 3x + 2 > 0$
$(x+2)(x+1) > 0$
$x > -1$ or $x < -2$



$x^2 + x - 2 < 0$
$(x+2)(x-1) < 0$
$-2 < x < 1$



The range of values fulfilling both inequalities is $-1 < x < 1$

H. Given that

$$\log_{10} 2 = 0.3010 \text{ to 4 d.p. and that } 10^{0.2} < 2$$

it is possible to deduce that

(a) 2^{100} begins in a 1 and is 30 digits long;
(b) 2^{100} begins in a 2 and is 30 digits long;
(c) 2^{100} begins in a 1 and is 31 digits long;
(d) 2^{100} begins in a 2 and is 31 digits long.

$\log_{10} 2 = 0.3010$
$100 \log_{10} 2 = 30.1$
$\log_{10} 2^{100} = 30.1$
$10^{30.1} = 2^{100}$
$2^{100} = 10^{30} \times 10^{0.1}$

$10^0 < 10^{0.1} < 10^{0.2} < 2$
$\therefore 10^{0.1}$ begins with 1
10^{30} is 1 followed by 30 zeros
$\therefore 10^{30}$ is 31 digits

$\therefore 2^{100}$ begins with 1 and is 31 digits

I. The power of x which has the greatest coefficient in the expansion of $\left(1+\frac{1}{2}x\right)^{10}$ is

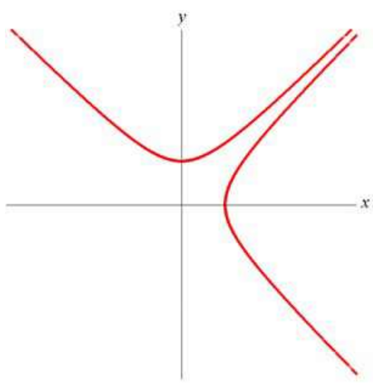
(a) x^2 (b) x^3 (c) x^5 (d) x^{10}

The coefficient of each power of x will equal $\frac{10!}{r!(10-r)!} \times \frac{1}{2^r}$,
where r is the power of x

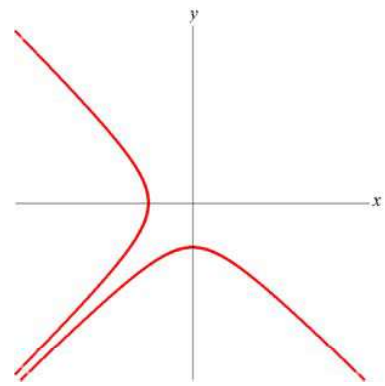
r	coefficient
2	$\frac{45}{4}$
3	15
5	$\frac{63}{8}$
10	$\frac{1}{2^{10}}$

$\therefore x^3$ has the greatest coefficient in the expansion of $\left(1+\frac{1}{2}x\right)^{10}$

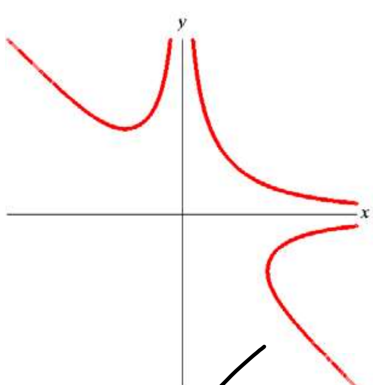
J. A sketch of the curve with equation $x^2y^2(x+y)=1$ is drawn in which of the following diagrams?



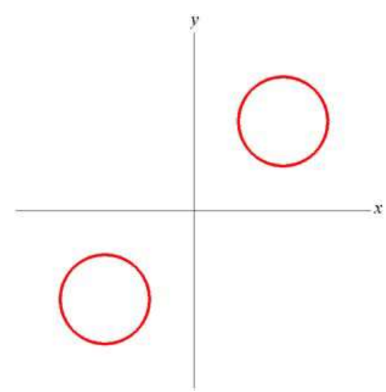
(a)



(b)



(c)



(d)

$x^2y^2(x+y)=1$

when $x=0$, the equation has no solutions, eliminating a and b

when $x=-1$

$y^2(-1+y)=1$

$-y^2+y^3=1$

$y^3=1+y^2$

as $1+y^2>0$

$y^3>0$

and $y>0$

this eliminates d (when $x<0$, $y<0$), while in c, when $x<0$, $y>0$

2. For **ALL APPLICANTS.**

(i) Show, with working, that

$$x^3 - (1 + \cos\theta + \sin\theta)\,x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)\,x - \sin\theta\cos\theta, \qquad (1)$$

equals

$$(x-1)\left(x^2 - (\cos\theta + \sin\theta)\,x + \cos\theta\sin\theta\right)$$

Deduce that the cubic in (1) has roots

$$1, \qquad \cos\theta, \qquad \sin\theta.$$

(ii) Give the roots when $\theta = \frac{\pi}{3}$.

(iii) Find all values of θ in the range $0 \leqslant \theta < 2\pi$ such that two of the three roots are equal.

(iv)What is the greatest possible difference between two of the roots, and for what values of θ in the range $0 \leqslant \theta < 2\pi$ does this greatest difference occur?

Show that for each such θ the cubic (1) is the same.

2i.

$$
\begin{array}{r l}
 & x^2 - (\cos\theta + \sin\theta)x + \cos\theta\sin\theta \\
(x-1) \,\big) & x^3 - (1+\cos\theta+\sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta \\
 & -x^3 \qquad +x^2 \\
 & -(\cos\theta+\sin\theta)x^2 + (\cos\theta\sin\theta+\cos\theta+\sin\theta)x - \sin\theta\cos\theta \\
 & +(\cos\theta+\sin\theta)x^2 \qquad -(\cos\theta+\sin\theta)x \\
 & (\cos\theta\sin\theta)x - \sin\theta\cos\theta \\
 & -(\cos\theta\sin\theta)x + \cos\theta\sin\theta \\
 & 0
\end{array}
$$

$(x-1)(x^2-(\cos\theta+\sin\theta)x+\cos\theta\sin\theta)=0$

$(x-1)(x-\cos\theta)(x-\sin\theta)=0$

roots are $1, \cos\theta, \sin\theta$

ii. $\theta = \frac{\pi}{3}$ roots are $1, \frac{1}{2}, \frac{\sqrt{3}}{2}$

iii. $\cos\theta = 1$ $\quad$ $\sin\theta = 1$ $\quad$ $\cos\theta = \sin\theta$

$\theta = 0$ $\quad$ $\theta = \frac{\pi}{2}$ $\quad$ $\tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$

iv. $-1 \leq \sin\theta \leq 1$ $\quad$ $-1 \leq \cos\theta \leq 1$

$1-(-1)=2$ greatest possible difference between roots is 2.

When $\sin\theta = -1$, $\cos\theta \neq 1$ and when $\cos\theta = -1$, $\sin\theta \neq 1$.

$\therefore$ greatest difference when $\sin\theta = -1$ or $\cos\theta = -1$,

$\theta = \frac{3\pi}{2}$ or $\theta = \pi$

When $\theta = \frac{3\pi}{2}$, $\sin\theta = -1$, $\cos\theta = 0$ and the cubic is $(x-1)(x+1)(x-0)$
$= x^3 - x$

When $\theta = \pi$, $\sin\theta = 0$, $\cos\theta = -1$ and the cubic is $(x-1)(x+1)(x-0)$
$= x^3 - x$

$\therefore$ for each θ, the cubic (1) is the same

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

In this question we shall consider the function $f(x)$ defined by

$$f(x) = x^2 - 2px + 3$$

where p is a constant.

(i) Show that the function $f(x)$ has one stationary value in the range $0 < x < 1$ if $0 < p < 1$, and no stationary values in that range otherwise.

In the remainder of the question we shall be interested in the smallest value attained by $f(x)$ in the range $0 \leqslant x \leqslant 1$. Of course, this value, which we shall call m, will depend on p.

(ii) Show that if $p \geqslant 1$ then $m = 4 - 2p$.

(iii) What is the value of m if $p \leqslant 0$?

(iv) Obtain a formula for m in terms of p, valid for $0 < p < 1$.

(v)Using the axes opposite, sketch the graph of m as a function of p in the range $-2 \leqslant p \leqslant 2$.

3i. $f(x) = x^2 - 2px + 3$
$f'(x) = 2x - 2p = 0$
$x = p$ at stationary point

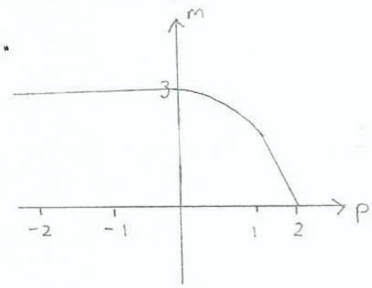
Since $0 < x < 1$, there is one stationary value if $0 < p < 1$ (as $x = p$).
If $p \geq 1$ or $p \leq 0$, there are no stationary values in the range $0 < x < 1$ (as x cannot equal p)

ii. $p \geq 1$, no stationary values in the range $0 \leq x \leq 1$ and p is positive, making the function decreasing in this range. $\therefore$ m occurs at $x = 1$
$m = 1^2 - 2 \times p \times 1 + 3$
$m = 4 - 2p$

iii. $p \leq 0$, no stationary values in the range $0 \leq x \leq 1$ and p is negative or zero, making the function increasing in this range. $\therefore$ m occurs at $x = 0$
$m = 0 - 2p \times 0 + 3$
$m = 3$

iv. $m = x^2 - 2px + 3$ $\quad x = p$
$m = p^2 - 2p^2 + 3$
$m = 3 - p^2$

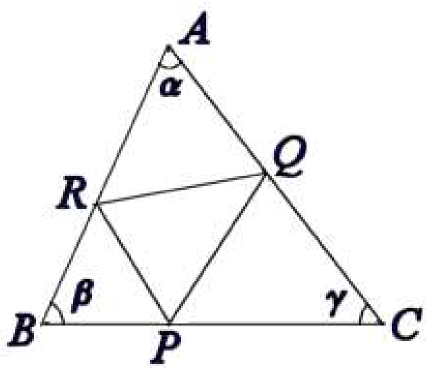
v.

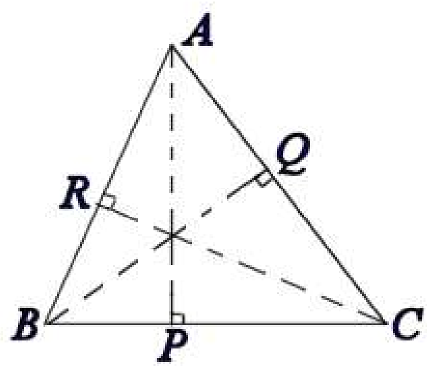


4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Maths & Computer Science **and** *Computer Science* **applicants should turn to page 14.**





A triangle ABC has sides BC, CA and AB of sides a, b and c respectively, and angles at A, B and C are α, β and γ where $0 \leqslant \alpha, \beta, \gamma \leqslant \frac{1}{2}\pi$.

(i) Show that the area of ABC equals $\frac{1}{2}bc\sin\alpha$.

Deduce the sine rule

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

(ii) The points P, Q and R are respectively the feet of the perpeniculars from A to BC, B to CA, and C to AB as shown.

Prove that

$$\text{Area of } PQR = \left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\right) \times (\text{Area of } ABC)\,.$$

(iii) For what triangles ABC, with angles α, β, γ as above, does the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

hold?

4i. $\sin\alpha = \frac{QB}{c}$

$c\sin\alpha = QB$

Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

Area of ABC $= \frac{1}{2} \times AC \times QB$

$= \frac{1}{2} \times b \times c\sin\alpha$

$= \frac{1}{2} bc\sin\alpha$

$$\frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

ii. Area of PQR = Area of ABC − (Areas of ARQ + BRP + PQC)

$= \frac{1}{2}bc\sin\alpha - \left(\frac{1}{2} \times AQ \times AR\sin\alpha + \frac{1}{2} \times BR \times BP \times \sin\beta + \frac{1}{2} \times PC \times CQ \sin\gamma\right)$

$\cos\alpha = \frac{AQ}{c}$ $\quad$ $\cos\alpha = \frac{AR}{b}$

$AQ = c\cos\alpha$ $\quad$ $AR = b\cos\alpha$

Area of ARQ $= \frac{1}{2} \times cb \times \cos^2\alpha \sin\alpha = \cos^2\alpha \times$ Area of ABC

In the same way, Area of BRP $= \cos^2\beta \times$ Area of ABC

and Area of PQC $= \cos^2\gamma \times$ Area of ABC

Area of PQR = Area of ABC $(1 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma))$

iii. Area of PQR = 0, which only occurs when ABC is right-angled.

5. For **ALL APPLICANTS.**

Songs of the Martian classical period had just two notes (let us call them x and y) and were constructed according to rigorous rules:

I. the sequence consisting of no notes was deemed to be a song (perhaps the most pleasant);

II. a sequence starting with x, followed by two repetitions of an existing song and ending with y was also a song;

III. the sequence of notes obtained by interchanging xs and ys in a song was also a song.

All songs were constructed using those rules.

(i) Write down four songs of length six (that is, songs with exactly six notes).

(ii) Show that if there are k songs of length m then there are $2k$ songs of length $2m+2$. Deduce that for each natural number there are 2^n songs of length $2^{n+1}-2$.

Songs of the Martian later period were constructed using also the rule:

IV. if a song ended in y then the sequence of notes obtained by omitting that y was also a song.

(iii) What lengths do songs of the later period have? That is, for which natural numbers n is there a song with exactly n notes? Justify your answer.

5i. $xxyxyy$, $xyxyxy \rightarrow$ Rule II
$yyxyxx$, $yxyxyx \rightarrow$ Rule III

ii. For each existing song of m length, two songs of $2m+2$ length can be created - one by rule II (x, then the existing song of m length twice, then y) and another by rule III (interchanging xs and ys in the song created by rule II), each leading to a unique song.
$\therefore$ if there are k songs of length m, there are $2k$ songs of length $2m+2$.

The first possible length would be 0. (length m is used to create songs of length $2m+2$)
The 2nd length $= 0\times 2 + 2 = 2$
The 3rd length $= 2\times 2 + 2 = 2^2 + 2$
The $(n+1)$th length $= 2^n + 2^{n-1} + \ldots + 2$ (geometric sum)
$$= \frac{2(2^n - 1)}{(2-1)} = 2^{n+1} - 2$$

There is 1 song of the 1st possible length. Each time the length increases, the number of songs double $\therefore$ at $(n+1)$th length, there would be 2^n songs, for all $n \geq 0$.

iii. All values of $n \geq 0$. Whenever a song ends in a y, it can be shortened (by 1) by omitting the y. (Rule IV) Whenever a song ends in an x, rule III can be applied, leading to a song ending with a y, which can then shortened as above.

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Alice, Bob and Charlie are well-known expert logicians.

(i) The King places a hat on each of their heads. Each of the logicians can see the others' hats, but not his or her own.

The King says "Each of your hats is either black or white, but you don't all have the same colour hat".

All four are honest, and all trust one another.

The King now asks Alice "Do you know what colour your hat is?".

Alice says "Yes, it's white".

What colour are the others' hats? [Hint: think about how Alice can deduce that her hat is white.]

(ii) The King now changes some of the hats, and again says "Each of your hats is either black or white, but you don't all have the same colour hat". He now asks Alice "Do you know what colour your hat is?".

Alice replies "No"

Can Bob work out what colour his hat is? Explain your answer. [Hint: what can Bob deduce from the fact that Alice can't tell what colour her hat is?]

(iii) The King now changes some of the hats, and then says "Each of your hats is either black or white. At least one of you has a white hat."

He now asks them all "Do you know what colour your hat is?". They all simultaneously reply "No".

What can you deduce about the colour of their hats? Explain your answer.

(iv) He again asks "Do you know what colour your hat is?" Alice says "No", but Bob and Charlie both say "Yes" (all three answer simultaneously).

What colour are their hats? Explain your answer.

6i. They do not all have the same colour hat. Therefore for Alice to know the colour of her hat, Bob and Charlie must both be wearing the same colour. As Alice says her hat is white, she must be able to see both Bob and Charlie wearing black hats.

ii. Alice does not know the colour of her hat, meaning Bob and Charlie must be wearing different coloured hats. Therefore Bob knows he has the opposite colour hat to Charlie.

iii. There must be at least two white hats as all three can see someone wearing a white hat and therefore all do not know the colour of their own hat. Hence one or no black hats

iv. Alice must be wearing a black hat, Bob and Charlie are wearing white hats. All have deduced that there are at least two white hats. Alice does not know the colour of her hat, but Bob and Charlie know theirs. This must be because they can see one black hat (Alice's) and therefore know they must be wearing a white hat each.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

The game of *Oxflip* is for one player and involves circular counters, which are white on one side and black on the other, placed in a grid. During a game, the counters are flipped over (changing between black and white side uppermost) following certain rules.

Given a particular size of grid and a set starting pattern of whites and blacks, the aim of the game is to reach a certain target pattern. Each "move" of the game is to flip over either a whole row or a whole column of counters (so one whole row or column has all its blacks swapped to whites and vice versa). For example, in a game played in a three-by-three square grid, if you are given the starting and target patterns

Start:		Target:	
	●○● ○●○ ●○●		○○○ ○○○ ○○○

a sequences of three moves to achieve the target is:

●○● ○●○ ●○●	flip first row $\Longrightarrow$	○●○ ○●○ ●○●	flip middle column $\Longrightarrow$	○○○ ○○○ ●●●	flip third row $\Longrightarrow$	○○○ ○○○ ○○○

There are many other sequences of moves which also have the same result.

(i) Consider the two-by-two version of the game with starting pattern

●○
○●

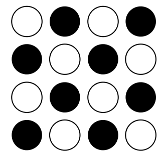
Draw, in the blank patterns below, the eight different target patterns (including the starting pattern) that it is possible to obtain.

●○ ○●	○● ○●	●● ●●	○○ ○○
○● ●○	●○ ●○	○○ ●●	●● ○○

What are the possible numbers of white counters that may be present in these target patterns?

The possible numbers of white counters are 0, 2, 4

(ii) In the four-by-four version of the game, starting with pattern

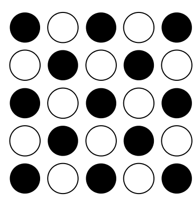


explain why it is impossible to reach a pattern with only one white counter.

[*Hint: don't try to write out every possible combination of moves.*]

In a row/column of 4 there is either an even number of black counters and an even number of white (in which case, a move would result also in an even number of each) or an odd number of black counters and an odd number of white counters (in which case, a move would result also in an odd number of each). In both cases, the overall number of white counters would remain even, as initially the number of white counters is even ∴ it is impossible for the number of white counters to reach 1, as 1 is an odd number and there is alwa s gn even number of white counters.

(iii) In the five-by-five game, explain why any sequence of moves which begins



and ends with an all-white pattern, must involve an odd number of moves. What is the least number of moves needed? Give reasons for your answer.

In a row/column of 5 there is either an even number of black counter and an odd number of white or an odd number of black counters and an even number of white. After each move, the number of white counters overall would change from odd to even or from even to odd. Therefore, for the number of white counters to change from 12 (an even number) at the beginning to 25 (an odd number) at the end, there must be an odd number of moves.

1 move is clearly not possible. 3 moves is also not possible (13 black counters cannot be grouped into 3 moves). 5 moves is possible (change rows 2 and 4, then columns 1, 3, 5). ∴ the least number of moves needed is 5.

End of Last Question